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Extended Conformal Algebra and Non-commutative Geometry in Particle Theory

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Abstract

We show how an off shell invariance of the massless particle action allows the construction of an extension of the conformal space-time algebra and induces a non-commutative space-time geometry in bosonic and supersymmetric particle theories.

1 Introduction

Relativistic particle theory may be looked at as a prototype theory for string theory and general relativity. Recently, there has been an intense research activity on the connection between string theory and non-commutative geometry. This is because it is believed that in quantum theories containing gravity, space-time must change its nature at distances comparable to the Planck scale. Quantum gravity has an uncertainty principle which prevents one from measuring positions to better accuracies than the Planck length [1]. These effects could then be modeled by a non-vanishing commutation relation between the space-time coordinates. Many developments in string and superstring theories were reached in this direction (see for instance [2], [3], [4] and cited references), and it is now believed that non-commutative geometry naturally arises from the string dynamics.

An approach to the study of the relations between string dynamics and particle dynamics, using the concept of space-time symmetry as an investigation tool, was presented in [5]. In particular, it was verified the existence of a very special string motion in the high energy limit of the theory. In this special motion, each point of the string moves as if it were a massless particle. The existence of such a string motion agrees with what is expected from gauge theory-string duality.

The above mentioned string motion opens the question of if we can find an indication that non-commutative geometry should also naturally emerge from the

massless relativistic particle dynamics. In this work we consider this question and show how we may use the special-relativistic ortogonality condition between the velocity and acceleration to induce a new invariance of the massless particle action. This induced symmetry may then be used to construct an off shell extension of the conformal space-time algebra in four dimensions and also to generate a transition to new space-time coordinates which obey non-vanishing commutation relations. The extended conformal algebra automatically reduces to the usual conformal algebra if the mass shell condition is imposed. The new commutation relations obey all Jacobi identities among the canonical variables, are preserved in the supersymmetric theory, and reduce to the usual commutation relations if the mass shell condition is imposed. The conclusion is that there exists an inertial frame in which the uncertainty introduced by two simultaneous position measurements is an off shell rotation in space-time. For clarity of our exposition of the subject, next section briefly reviews the concept of conformal vector fields.

2 Conformal Vector Fields

Consider the Euclidean flat space-time vector field

$$\hat{R}(\epsilon) = \epsilon^{\mu}\partial_{\mu} \tag{2.1}$$

such that

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \delta_{\mu\nu}\partial.\epsilon \tag{2.2}$$

The vector field $\hat{R}(\epsilon)$ gives rise to the coordinate transformation

$$\delta x^{\mu} = \hat{R}(\epsilon)x^{\mu} = \epsilon^{\mu} \tag{2.3}$$

The vector field (2.1) is known as the Killing vector field and ϵ^{μ} is known as the Killing vector. One can show that the most general solution for equation (2.2) in a four-dimensional space-time is

$$\epsilon^{\mu} = \delta x^{\mu} = a^{\mu} + \omega^{\mu\nu}x_{\nu} + \alpha x^{\mu} + (2x^{\mu}x^{\nu} - \delta^{\mu\nu}x^2)b_{\nu} \tag{2.4}$$

The vector field $\hat{R}(\epsilon)$ for the solution (2.4) can then be written as

$$\hat{R}(\epsilon) = a^{\mu}P_{\mu} - \frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} + \alpha D + b^{\mu}K_{\mu} \tag{2.5}$$

where

$$P_{\mu} = \partial_{\mu} \tag{2.6}$$

$$M_{\mu\nu} = x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} \tag{2.7}$$

$$D = x^{\mu}\partial_{\mu} \tag{2.8}$$

$$K_\mu = (2x_\mu x^\nu - \delta_\mu^\nu x^2)\partial_\nu \tag{2.9}$$

P_μ generates translations, $M_{\mu\nu}$ generates rotations, D generates dilatations and K_μ generates conformal transformations in space-time. The generators of the vector field $\hat{R}(\epsilon)$ obey the commutator algebra

$$[P_\mu, P_\nu] = 0 \tag{2.10a}$$

$$[P_\mu, M_{\nu\lambda}] = \delta_{\mu\nu} P_\lambda - \delta_{\mu\lambda} P_\nu \tag{2.10b}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \delta_{\nu\lambda} M_{\mu\rho} + \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\lambda} M_{\nu\rho} \tag{2.10c}$$

$$[D, D] = 0 \tag{2.10d}$$

$$[D, P_\mu] = -P_\mu \tag{2.10e}$$

$$[D, M_{\mu\nu}] = 0 \tag{2.10f}$$

$$[D, K_\mu] = K_\mu \tag{2.10g}$$

$$[P_\mu, K_\nu] = 2(\delta_{\mu\nu} D - M_{\mu\nu}) \tag{2.10h}$$

$$[M_{\mu\nu}, K_\lambda] = \delta_{\nu\lambda} K_\mu - \delta_{\lambda\mu} K_\nu \tag{2.10i}$$

$$[K_\mu, K_\nu] = 0 \tag{2.10j}$$

The commutator algebra (2.10) is the conformal space-time algebra in four dimensions. Notice that the commutators (2.10a-2.10c) correspond to the Poincaré algebra. The Poincaré algebra is a sub-algebra of the conformal algebra. Let us now see how we can extend the conformal algebra, and how this extended algebra is related to a non-commutative space-time geometry.

3 Relativistic Particles

A relativistic particle describes is space-time a one-parameter trajectory $x^\mu(\tau)$. A possible form of the action is the one proportional to the arc length traveled by the particle and given by

$$S = -m \int ds = -m \int d\tau \sqrt{-\dot{x}^2} \tag{3.1}$$

In this work we choose τ to be the particle's proper time, m is the particle's mass and $ds^2 = -\delta_{\mu\nu} dx^\mu dx^\nu$. A dot denotes derivatives with respect to τ and we use units in which $\hbar = c = 1$.

Action (3.1) is obviously inadequate to study the massless limit of the theory and so we must find an alternative action. Such an action can be easily computed by treating the relativistic particle as a constrained system. In the transition to the Hamiltonian formalism action (3.1) gives the canonical momentum

$$p_\mu = \frac{m}{\sqrt{-\dot{x}^2}} \dot{x}_\mu \tag{3.2}$$

and this momentum gives rise to the primary constraint

$$\phi = \frac{1}{2}(p^2 + m^2) = 0 \tag{3.3}$$

We follow Dirac's [6] convention that a constraint is set equal to zero only after all calculations have been performed. The canonical Hamiltonian corresponding to action (3.1), $H = p.\dot{x} - L$, identically vanishes. This is a characteristic feature of reparametrization-invariant systems. Dirac's Hamiltonian for the relativistic particle is then

$$H_D = H + \lambda\phi = \frac{1}{2}\lambda(p^2 + m^2) \tag{3.4}$$

where $\lambda(\tau)$ is a Lagrange multiplier. The Lagrangian that corresponds to (3.4) is

$$\begin{aligned} L &= p.\dot{x} - H_D \\ &= p.\dot{x} - \frac{1}{2}\lambda(p^2 + m^2) \end{aligned} \tag{3.5}$$

Solving the equation of motion for p_μ that follows from (3.5) and inserting the result back in it, we obtain the particle action

$$S = \int d\tau \left(\frac{1}{2}\lambda^{-1}\dot{x}^2 - \frac{1}{2}\lambda m^2\right) \tag{3.6}$$

The great advantage of action (3.6) is that it has a smooth transition to the $m = 0$ limit.

Action (3.6) is invariant under the Poincaré transformations

$$\delta x^\mu = a^\mu + \omega^\mu_\nu x^\nu \tag{3.7a}$$

$$\delta\lambda = 0 \tag{3.7b}$$

Invariance of action (3.6) under transformation (3.7a) implies that we can construct a space-time vector field corresponding to the first two generators in the right of equation (2.5). These generators realize the Poincaré algebra (2.10a-2.10c).

Now we make a transition to the massless limit. This limit is described by the action

$$S = \frac{1}{2}\int d\tau \lambda^{-1}\dot{x}^2 \tag{3.8}$$

The canonical momentum conjugate to x^μ is

$$p_\mu = \frac{1}{\lambda}\dot{x}_\mu \tag{3.9}$$

The canonical momentum conjugate to λ identically vanishes and this is a primary constraint, $p_\lambda = 0$. Constructing the canonical Hamiltonian, and requiring the stability of this constraint, we are led to the mass shell condition

$$\phi = \frac{1}{2}p^2 = 0 \tag{3.10}$$

Let us now study which space-time symmetries are present in this limit. Being the $m = 0$ limit of (3.6), action (3.8) is also invariant under transformation (3.7). The massless action (3.8) however has a larger set of space-time invariances. It is also invariant under the scale transformation

$$\delta x^\mu = \alpha x^\mu \tag{3.11a}$$

$$\delta\lambda = 2\alpha\lambda \tag{3.11b}$$

where α is a constant, and under the conformal transformation

$$\delta x^\mu = (2x^\mu x^\nu - \delta^{\mu\nu}x^2)b_\nu \tag{3.12a}$$

$$\delta\lambda = 4\lambda x.b \tag{3.12b}$$

where b_μ is a constant vector. Invariance of action (3.8) under transformations (3.7a), (3.11a) and (3.12a) then implies that the full conformal field (2.5) can be defined in the massless sector of the theory.

It is convenient at this point to relax the mass shell condition (3.10). This is because the massless particle will be off mass shell in the presence of interactions [4]. We may now use the fact that we are dealing with a special-relativistic system. Special relativity has the characteristic kinematical feature that the relativistic velocity is always ortogonal to the relativistic acceleration (see, for instance, [7]). Then, as a consequence of the fact that action (3.8) is a special-relativistic action, it is invariant under the transformation

$$x^\mu \to \tilde{x}^\mu = \exp\{\beta(\dot{x}^2)\}x^\mu \tag{3.13a}$$

$$\lambda \to \exp\{2\beta(\dot{x}^2)\}\lambda \tag{3.13b}$$

where β is an arbitrary function of $\dot{x}^2$. We emphasize that although the ortogonality condition must be used to get the invariance of action (3.8) under

transformation (3.13), this condition is not an external ingredient in the theory. In fact, the ortogonality between the relativistic velocity and acceleration is an unavoidable condition here, it is an imposition of special relativity.

It may be pointed out that since the relativistic ortogonality condition is used to make the massless action invariant under (3.13), these transformations may not be a true invariance of the action, being at most a symmetry of the equations of motion. If we compute the classical equation of motion for x^μ that follows from the massive action (3.6) we will find that

$$\frac{d}{d\tau}(\frac{1}{\lambda}\dot{x}^\mu) = 0 \tag{3.14}$$

This equation of motion is identical to the one that follows from the massless action (3.8). We may say that at the classical level the massive and the massless relativistic particles are governed by the same relativistic dynamics. Now, while the classical equation of motion (3.14) is invariant under transformation (3.13) when the ortogonality condition is used, the massive action (3.6) is not. Transformation (3.13) is a symmetry of the equations of motion only in the massive theory. In the massless theory it is a symmetry of the equations of motion and of the action.

Now, invariance of the massless action under transformations (3.13) means that infinitesimally we can define the scale transformation

$$\delta x^\mu = \alpha\beta(\dot{x}^2)x^\mu \tag{3.15}$$

where α is the same constant that appears in equations (2.4) and (2.5). These transformations then lead to the existence of a new type of dilatations. These new dilatations manifest themselves in the fact that the vector field D of equation (2.8) can be changed according to

$$D = x^\mu\partial_\mu \to D^* = x^\mu\partial_\mu + \beta(\dot{x}^2)x^\mu\partial_\mu = D + \beta D \tag{3.16}$$

In fact, because all vector fields in equation (2.5) involve partial derivatives with respect to x^μ , and β is a function of $\dot{x}^\mu$ only, we can also introduce the generators

$$P^*_\mu = P_\mu + \beta P_\mu \tag{3.17}$$

$$M^*_{\mu\nu} = M_{\mu\nu} + \beta M_{\mu\nu} \tag{3.18}$$

$$K^*_\mu = K_\mu + \beta K_\mu \tag{3.19}$$

and define the new space-time vector field

$$V^*_0 = a^\mu P^*_\mu - \frac{1}{2}\omega^{\mu\nu}M^*_{\mu\nu} + \alpha D^* + b^\mu K^*_\mu \tag{3.20}$$

The generators of this vector field obey the algebra

$$[P^*_\mu, P^*_\nu] = 0 \tag{3.21a}$$

$$[P^*_\mu, M^*_{\nu\lambda}] = (\delta_{\mu\nu} P^*_\lambda - \delta_{\mu\lambda} P^*_\nu) + \beta(\delta_{\mu\nu} P^*_\lambda - \delta_{\mu\lambda} P^*_\nu) \tag{3.21b}$$

$$[M^*_{\mu\nu}, M^*_{\lambda\rho}] = (\delta_{\nu\lambda} M^*_{\mu\rho} + \delta_{\mu\rho} M^*_{\nu\lambda} - \delta_{\nu\rho} M^*_{\mu\lambda} - \delta_{\mu\lambda} M^*_{\nu\rho})$$

$$+\beta(\delta_{\nu\lambda} M^*_{\mu\rho} + \delta_{\mu\rho} M^*_{\nu\lambda} - \delta_{\nu\rho} M^*_{\mu\lambda} - \delta_{\mu\lambda} M^*_{\nu\rho}) \tag{3.21c}$$

$$[D^*, D^*] = 0 \tag{3.21d}$$

$$[D^*, P^*_\mu] = -P^*_\mu - \beta P^*_\mu \tag{3.21e}$$

$$[D^*, M^*_{\mu\nu}] = 0 \tag{3.21f}$$

$$[D^*, K^*_\mu] = K^*_\mu + \beta K^*_\mu \tag{3.21g}$$

$$[P^*_\mu, K^*_\nu] = 2(\delta_{\mu\nu} D^* - M^*_{\mu\nu}) + 2\beta(\delta_{\mu\nu} D^* - M^*_{\mu\nu}) \tag{3.21h}$$

$$[M^*_{\mu\nu}, K^*_\lambda] = (\delta_{\lambda\nu} K^*_\mu - \delta_{\lambda\mu} K^*_\nu) + \beta(\delta_{\lambda\nu} K^*_\mu - \delta_{\lambda\mu} K^*_\nu) \tag{3.21i}$$

$$[K^*_\mu, K^*_\nu] = 0 \tag{3.21j}$$

Notice that the vanishing brackets of the conformal algebra (2.10) are preserved as vanishing in the above algebra, but the non-vanishing brackets of the conformal algebra now have linear and quadratic contributions from the arbitrary function $\beta(\dot{x}^2)$. Algebra (3.21) is an off shell extension of the conformal algebra (2.10).

Now consider the commutator structure induced by transformation (3.13a). We assume the usual commutation relations between the canonical variables, $[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0$, $[x_\mu, p_\nu] = i\delta_{\mu\nu}$. Taking $\beta(\dot{x}^2) = \beta(\lambda^2 p^2)$ in transformation (3.13a) and transforming the p_μ in the same manner as the x_μ, we find that the new transformed canonical variables $(\tilde{x}_\mu, \tilde{p}_\mu)$ obey the commutators

$$[\tilde{p}_\mu, \tilde{p}_\nu] = 0 \tag{3.22}$$

$$[\tilde{x}_\mu, \tilde{p}_\nu] = i\delta_{\mu\nu}(1+\beta)^2 + (1+\beta)[x_\mu, \beta]p_\nu \tag{3.23}$$

$$[\tilde{x}_\mu, \tilde{x}_\nu] = (1+\beta)\{x_\mu[\beta, x_\nu] - x_\nu[\beta, x_\mu]\} \tag{3.24}$$

written in terms of the old canonical variables. These commutators obey the non trivial Jacobi identities $(\tilde{x}_\mu, \tilde{x}_\nu, \tilde{x}_\lambda) = 0$ and $(\tilde{x}_\mu, \tilde{x}_\nu, \tilde{p}_\lambda) = 0$. They also reduce to the usual canonical commutators when $\beta(\lambda^2 p^2) = 0$.

The simplest example of this geometry is the case when $\beta(\lambda^2 p^2) = \lambda^2 p^2$. The new positions then satisfy

$$[\tilde{x}_\mu, \tilde{x}_\nu] = -2i\lambda^2 M^*_{\mu\nu} \tag{3.25}$$

where $M^*_{\mu\nu}$ is the extended off shell operator of Lorentz rotations given by equation (3.18). The commutator (3.25) satisfies

$$\int \mathbf{Tr}[\tilde{x}_\mu, \tilde{x}_\nu] = 0 \tag{3.26}$$

as is the case for a general non-commutative algebra [8]. From equation (3.25) we may say that there exists an inertial frame in which the uncertainty introduced by two simultaneous position measurements is an off shell rotation in space-time.

Finally we consider the case of the massless superparticle. It is described by the action [9]

$$S = \frac{1}{2}\int d\tau \lambda^{-1}(\dot{x}^\mu - i\bar{\theta}\Gamma^\mu\dot{\theta})^2 \tag{3.27}$$

where θ_α is a space-time spinor and $\Gamma^\mu_{\alpha\beta}$ are Dirac matrices. The canonical momentum conjugate to x^μ is

$$p_\mu = \frac{1}{\lambda}(\dot{x}_\mu - i\bar{\theta}\Gamma_\mu\dot{\theta}) = \frac{1}{\lambda}Z_\mu \tag{3.28}$$

where we introduced the supersymmetric [9] variable $Z^\mu = \dot{x}^\mu - i\bar{\theta}\Gamma^\mu\dot{\theta}$. As in the bosonic case, the primary constraint $p_\lambda = 0$ leads to the mass shell condition $\phi = \frac{1}{2}p^2 = 0$.

If we extend the calculations in [7] to the case of the massless superparticle we will find that the relation

$$Z.\frac{dZ}{d\tau} = 0 \tag{3.29}$$

must hold. Equation (3.29) is the supersymmetric extension of the special-relativistic condition of ortogonality between velocity and acceleration. Again, condition (3.29) is an imposition of the supersymmetric relativistic theory, and not an artificially introduced external ingredient.

Relaxing again the mass shell condition, we find that the massless superparticle action (3.27) must be invariant under the transformation

$$x^\mu \rightarrow \tilde{x}^\mu = \exp\{\beta(Z^2)\}x^\mu \tag{3.30a}$$

$$\theta_\alpha \rightarrow \tilde{\theta}_\alpha = \exp\{\frac{1}{2}\beta(Z^2)\}\theta_\alpha \tag{3.30b}$$

$$\lambda \rightarrow \exp\{2\beta(Z^2)\}\lambda \tag{3.30c}$$

where β is now an arbitrary function of Z^2. In the canonical formalism $\beta(Z^2) = \beta(\lambda^2 p^2)$. Since the bosonic momentum p_μ commutes with the fermionic canonical variables θ_α and π_α , this leaves invariant the anti-commutation relations

between the fermionic variables but change the bosonic ones in the same way as (3.22-3.24). The commutator structure we found for the bosonic massless particle is thus preserved in the supersymmetric massless theory.

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